Piecewise & Taylor Series

Ally Macdonald

December 15, 2021

1 Introduction

An APO is an anonymous piecewise object that minimally satisifies a certain set of conditions, and is most easily formulated as $\phi^\star = \mu + [\phi] \cdot r$.

A system of APOs is like a system of equations, in that you want an object to not only satisfy a single APO, but multiple distinct APOs; that is, there exists a $\phi^\star$ for which:

$$\begin{aligned} \phi^\star &= \mu_1 + [\phi]_1 \cdot r_1 \\ \phi^\star &= \mu_2 + [\phi]_2 \cdot r_2 \\ &\vdots \\ \phi^\star &= \mu_n + [\phi]_n \cdot r_n \end{aligned}$$

This system is generally more difficult than linear systems depending on the operation(s) under which the system is formed.

The systems in this document that are generated are formed under differentiation and are fairly straightforward to solve with the help of recursion and induction. This technique was realised during the derivation of smoothstep functions in a similar manner (instead using definite values with 2 particular points of interest).

2 Derivation

2.1 Setup

Let $\{a_n\}$ be a sequence for which $f^{(n)}(a) = a_n$, where f is an n-times differentiable function near $x = a$.

We note we have the following system of APOs, where $F_{0,k}(x)$ are differentiable functions near a:

$$\begin{aligned} f(x) &= \begin{cases} a_0 & x = a \\ \star & \star \end{cases} & &= a_0 + (x-a)F_{0,0}(x) \\ f'(x) &= \begin{cases} a_1 & x = a \\ \star & \star \end{cases} & &= a_1 + (x-a)F_{0,1}(x) \\ &\vdots \\ f^{(n)}(x) &= \begin{cases} a_n & x = a \\ \star & \star \end{cases} & &= a_n + (x-a)F_{0,n}(x) \end{aligned}$$

More generally, we write for $m \geq 1$:

$$\begin{aligned} f(x) &= a_0 + (x-a)F_{0,0}(x) \\ f^{(m)}(x) &= a_m + (x-a)F_{0,m}(x) \end{aligned}$$

From the first equation, it can be inductively shown that $f^{(m)}(x) = mF_{0,0}^{(m-1)}(x) + F_{0,0}^{(m)}(x)$. Equating this equation with the above and letting $x = a$, we have for $m \geq 1$ that:

$$F_{0,0}^{(m-1)}(a) = \frac{a_m}{m}$$

We apply the same technique, as before, to give us, for new functions:

$$\begin{aligned} F_{0,0}(x) &= a_1 + (x-a)F_{1,1}(x) \\ F_{0,0}^{(m-1)}(x) &= \frac{a_m}{m} + (x-a)F_{1,m}(x) \end{aligned}$$

Notice that this system is nearly equivalent to the first system of equations, with two noticeable changes:

- $m \geq 1$ and so there is one less equation in the system than before; fewer derivatives.
- The leading term $\frac{a_m}{m}$ has differed from our original a_m and will be affected by the derivatives. Convince yourself that that this yields a factorial on the leading term by applying the technique from before to the new system above.

2.2 General System (of Systems)

From our above setup, we can set up an induction for which we can observe $0 \leq k < n$:

$$\begin{aligned} F_{k,k}(x) &= \frac{a_{k+1}}{(k+1)!} + (x-a)F_{k+1,k+1}(x) \\ F_{k,k}^{(m-k-1)}(x) &= \frac{a_m}{m(m-1)\dots(m-k)} + (x-a)F_{k+1,m}(x) \end{aligned}$$

Since we know this holds for $k = 0$, as this case satisfies the above, let us suppose for $0 \leq k < n$ we have the above hold true for the k case.

From the first equation, we inductively show that:

$$F_{k,k}^{(m-k-1)}(x) = (m-k-1)F_{k+1,k+1}^{(m-k-2)}(x) + (x-a)F_{k+1,k+1}^{(m-k-1)}(x)$$

Equating with the second equation as before, we have then for $x = a$:

$$F_{k+1,k+1}^{(m-k-2)}(a) = \frac{a_m}{m(m-1)\dots(m-k-1)}$$

This yields the following:

$$\begin{aligned} F_{k+1,k+1}^{(m-k-2)}(x) &= \frac{a_m}{m(m-1)\dots(m-k-1)} + (x-a)F_{k+2,m}(x) \\ F_{k+1,k+1}(x) &= \frac{a_{k+2}}{(k+2)!} + (x-a)F_{k+2,k+2}(x) \end{aligned}$$

Where the latter equation is derived from the former; $m = k+2$, hence proven as required.

2.3 Expanding

From the following:

$$f(x) = a_0 + (x-a)F_{0,0}(x)$$
$$F_{k,k}(x) = \frac{a_{k+1}}{(k+1)!} + (x-a)F_{k+1,k+1}(x)$$

We can recursively expand (or you can do more induction) to give:

$$f(x) = a_0 + \frac{a_1}{1!}(x-a) + \frac{a_2}{2!}(x-a)^2 + \cdots + \frac{a_n}{n!}(x-a)^n + F_{n,n}(x)(x-a)^{n+1}$$

For a minimal polynomial that satisfies all n derivatives, one might set $F_{n+1,n+1}(x) = 0$ as per standard interpolation/APO techniques.

3 Polynomial Similarity Theorem

3.1 Claim

All polynomials that share an arbitrary number of points and up to the nth derivatives at those points can be written in the form:

$$\phi^\star(x) = \phi(x) + [\phi(x)]^{n+1}r(x)$$

Where $\phi(x)$ is a given polynomial that satisfies the points and derivatives and $[\phi(x)]$ is a polynomial with zeroes at those points. $r(x)$ is an arbitrary polynomial.

3.2 Proof

We consider all polynomials of the form $\phi^\star(x) = \phi(x) + [\phi(x)]r(x)$.

We note that the nth derivative of this polynomial has equation:

$$\phi^{\star(n)}(x) = \phi^{(n)}(x) + [\phi(x)]^{(n)}r(x) + [\phi(x)]^{(n-1)}(\dots) + \cdots + [\phi(x)](\dots)$$

And also that we want that $\phi^{\star(n)}(x) = \phi^{(n)}(x)$ when:

$$[\phi(x)] = 0, [\phi(x)]' = 0, \dots, [\phi(x)]^{(n-1)} = 0$$

That is, firstly, the nth derivative of $\phi(x)$ at the points is shared by $\phi^\star(x)$ *and* all derivatives up to the nth derivative are shared, which is when each root polynomial is 0 by definition.

We therefore have the equation $[\phi(x)]^{(n)}r(x) = 0$.

3.2.1 Case 1

From the above equation, we consider when $r(x)$ vanishes, but not for all x (which is a trivial solution). $r(x)$ vanishes iff all conditions above are met; that is $[\phi(x)] = 0, [\phi(x)]' = 0, \dots, [\phi(x)]^{(n-1)} = 0$.

Equivalently, this means that if we consider $[\lambda(x)]$ to have all of the zeroes as above, we can write, for arbitrary polynomials $p(x)$ and $s(x)$:

$$[\lambda(x)] = p(x) \prod_{k=0}^{n-1} [\phi(x)]^{(k)}$$
$$r(x) = [\lambda(x)] + [[\lambda(x)]]\, s(x)$$

Noting furthermore that the decider of $[\lambda(x)]$ shares zeroes with $[\lambda(x)]$ itself, then, for yet another arbitrary polynomial $q(x)$:

$$r(x) = q(x)[\lambda(x)]$$

We now consider the derivatives of the respective deciders in $[\lambda(x)]$. Note that for all m from 0 up until $n-2$, the $m+1$ derivative of the decider shares points with the m derivative of the decider;

$$[\phi(x)]^{(m+1)} = [\phi(x)]^{(m)} + [[\phi(x)]^{(m)}]\alpha_m(x)$$
$$\implies [\phi(x)]^{(m+1)} = [\phi(x)]^{(m)}\lambda_m(x)$$

Solving the recurrence relation, we get that $[\phi(x)]^{(m+1)} = [\phi(x)]m(x)$ (yet an another arbitrary polynomial). Alternatively, one could have (weakly) noted that each derivative of the decider shares zeroes with the decider itself, and achieved the same result.

Using the above result, we have that:

$$r(x) = l(x)[\phi(x)]^n$$

Hence we have that $\phi^\star(x) = \phi(x) + [\phi(x)]^{n+1}l(x)$, for some arbitrary polynomial $l(x)$, which can be 'relabelled' as $r(x)$.

3.2.2 Case 2

We consider now cases when $[\phi(x)]^{(n)}$ vanishes (sometimes).

Suppose $[\phi(x)]^{(n)}$ vanishes at a subset of the arbitrary chosen points, but not others, then $r(x)$ vanishes at those other points. Then $[\phi(x)]^{(n)}r(x)$ shares zeroes with $[\phi(x)]$ (this can be shown using the same reasoning as in the previous case).

$$[\phi(x)]^{(n)}r(x) = [\phi(x)]s(x)$$

Since this ODE in $[\phi(x)]$ needs to be solvable for all non-zero polynomials $r(x), s(x)$, and for general $n \in \mathbb{Z}^+$, then the only viable solution is for when $[\phi(x)] = 0$ which gives the trivial solution (also, $[\phi(x)]$ is a polynomial).

Note: For non-polynomials, there may be solutions to the above. This gives rise to a weaker but effective result that does not require our functions to be polynomials, but also does not ensure all functions can be written in a specific form (e.g. for definite $[\phi(x)]$) as given by the theorem.

3.3 Corollary

Suppose two polynomials share infinitely many derivatives at an arbitrary non-zero number of points. Then those two polynomials must be identical everywhere.

$$\phi^{\star}(x) = \phi(x) + \lim_{n\to\infty} [\phi(x)]^{n+1} r(x)$$

For $[\phi(x)]$ to be well-defined and continuous everywhere, it means either $[\phi(x)]$ or $r(x)$ is identically 0. Note that this is also a trivial solution we found in the proof for polynomial derivatives above. In any case,

$$\phi^{\star}(x) = \phi(x)$$

Hence proven.

4 Final Result

Suppose then we have a function, $f(x)$, sufficiently represented by the series:

$$f(x) = \sum_{k=0}^{n} \frac{a_k}{k!}(x-a)^k + \cancelto{0}{F(x)(x-a)^{n+1}}$$

By the above corollary, since the Taylor series encodes all derivatives at $x = a$ for $n \to \infty$, then if $a_k = g^{(k)}(a)$ for sufficiently convergent and differentiable function $g(x)$, we have for all x:

$$f(x) = g(x)$$

Hence:

$$f(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!}(x-a)^k$$

5 Remainder Term

Consider the trailing term from our original result sum:

$$P(x) = F_{n,n}(x)(x-a)^{n+1}$$

Notice that by our polynomial similarity theorem, if $F_{n,n}(x)$ is itself a polynomial, then this term in fact shares n derivatives (and value) at $x = a$ with the zero polynomial. That is, $P(a) = 0,\ P'(a) = 0,\ \ldots,\ P^{(n)}(a) = 0$. By our weaker theorem, this can be extended to non-polynomials, but limits the functions which can be written as such.

Our series converges on some interval I if $P(x) \to 0$ as $n \to \infty$. Suppose our function is $n+1$ times differentiable, then $P(x)$ can be written:

$$P(x) = \lim_{n\to\infty} \frac{g^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} = 0$$

Since $F_{n,n}(x)(x-a)^{n+1} \approx F_{n+1,n+1}(x-a)^{n+2}$ for large n. As such, we now have a test for convergence as given by the limit above.