Piecewise-Constructed Functions

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1 Piecewise via Function Composition

There are a class of piecewise functions that can be formed via the composition of two traditionally non-piecewise functions. Incidentally, these compositions form some well known piecewise functions.

We begin by letting $U, V, D \subset \mathbb{R}$, $U \subseteq D$, be sets and $F : D \to V$ a function. We define the function $f = F|_U$ (the restriction of F to U), so that there exists a function $g : V \to U$ for which:

$$\begin{aligned} (g \circ f)(x) &= x, & x &\in U \\ (f \circ g)(x) &= x, & x &\in V \end{aligned}$$

In other words, $f(x)$ is a bijective function. We now wish to find the function $(g \circ F)(x)$; namely, an extension of the left inverse of f, over the reals. We also note immediately that $(F \circ g)(x) = x$ as $x \in V$.

We begin by first defining the function $p : D \to U$, $p(x) = (g \circ F)(x)$. Using the fact that $(F \circ g)(x) = x$, we have that $(F \circ p)(x) = F(x)$. If this equation can be solved for $p(x)$, then we have that (in most general form):

$$p(x) = \{p_i(x), \quad U_i \mid i \in I\}$$

Where $p_i(x)$ is a solution of the above on some domain, and U_i defines that domain. To find U_i, we note that $p(x) \in U \implies p_i(x) \in U$. Therefore:

$$p(x) = \{p_i(x), \quad p_i(x) \in U \mid i \in I\}$$

To verify that this is our solution, we ensure that we have all solutions for $p(x)$; more specifically:

$$\bigvee_{i \in I} (p_i(x) \in U) \iff x \in D$$

Alternatively, using Iverson bracket notation;

$$\sum_{i \in I} [p_i(x) \in U] = [x \in D]$$

1.1 Absolute Value

The absolute value function is special in that it can be 'created' in infinitely many different ways via a simple class of functions: the even functions with range $[0, \infty)$.

Let $F : \mathbb{R} \to [0, \infty)$ be an even function. Then we let $f = F|_{[0,\infty)}$ be a function for which there exists $g : [0, \infty) \to [0, \infty)$ such that $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$ for $x \in [0, \infty)$.

Let $p(x) = (g \circ F)(x)$, then $F(p(x)) = F(x)$. For $x \geq 0$, we have $f(p(x)) = f(x) \iff p(x) = x$, by bijectivity of f. For $x \leq 0$, by the evenness of F we have $F(p(x)) = F(-x)$, and by the same argument gives $p(x) = -x$.

Therefore:

$$p(x) = (g \circ F)(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases} = |x|$$

2 $\sin(x)$ and $\arcsin(x)$

The function $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}]$ has an inverse function, $\arcsin : [-1, 1]$, for which both $\sin(\arcsin(x)) = x$ and $\arcsin(\sin(x)) = x$ are true. However, we wonder what would 'happen' if the domain of sin were extended to the reals. We know that arcsin ceases to be an inverse of sin over the reals. But more specifically, what does $\arcsin(\sin(x))$ evaluate to, and what does $\sin(\arcsin(x))$ evaluate to?

2.1 $f(x) = \sin(\arcsin(x))$

Firstly, we consider that any function $f(x) = \sin(\arcsin(x))$ must have a domain $x \in [-1, 1]$ and that $f(x) \in [-1, 1]$ also. We could stop here, but for the sake of consistency I'll keep going with this argument.

To consider how $f(x)$ is constructed, we look for solutions in terms of x, ideally without a composition of inverses. To do this, we take $\arcsin(f(x)) = \arcsin(x)$ which gives $f(x) = x$, noting that $\sin(\arcsin(x)) = x$ over our domain.

To further verify that $f(x) = x$ is the only piece, we take that $f(x) \in [-1, 1]$ and substitute the piece in; namely, $x \in [-1, 1]$ which is just our domain.

Therefore:

$$f : [-1, 1], f(x) = x$$

(Hint: this is the boring one)

2.2 $f(x) = \arcsin(\sin(x))$

To establish what exactly this function is, we need to solve for $f(x)$ without a composition of inverses, a similar process to the previous function. However, this time, we can't use $\arcsin(x)$; namely, because we don't know that for $x \in \mathbb{R}$, that $\arcsin(\sin(x)) = x$: that's what we're trying to (not) show.

Firstly, consider $\sin(f(x)) = \sin(x)$ which gives $\sin(f(x)) - \sin(x) = 0$.

We now note that $\sin(a + b) + \sin(a - b) = 2\sin(a)\cos(b)$. If we let $a = \frac{f(x)-x}{2}$ and $b = \frac{f(x)+x}{2}$ we get the following:

$$\sin(f(x)) - \sin(x) = 2\sin\left(\frac{f(x) - x}{2}\right)\cos\left(\frac{f(x) + x}{2}\right) = 0$$

Hence, we have solutions:

$$\frac{1}{2}(f(x) - x) = m\pi \implies f(x) = 2m\pi + x \qquad m \in \mathbb{Z} \tag{1}$$

$$\frac{1}{2}(f(x) + x) = \frac{(2n + 1)\pi}{2} \implies f(x) = (2n + 1)\pi - x \qquad n \in \mathbb{Z} \tag{2}$$

Then we use the fact that $-\frac{\pi}{2} \leq f(x) \leq \frac{\pi}{2}$ to establish domains for our respective pieces:

$$-\frac{\pi}{2} \leq 2m\pi + x \leq \frac{\pi}{2} \implies -\frac{\pi}{2}(4m + 1) \leq x \leq -\frac{\pi}{2}(4m - 1) \tag{3}$$

$$-\frac{\pi}{2} \leq (2n + 1)\pi - x \leq \frac{\pi}{2} \implies (4n + 1)\frac{\pi}{2} \leq x \leq (4n + 3)\frac{\pi}{2} \tag{4}$$

We know that this function is continuous; we could write it piecewise now as the following function:

$$f(x) = \left\{2m\pi + x, \quad x \in \left[-\frac{\pi}{2}(4m+1), -\frac{\pi}{2}(4m-1)\right] \mid m \in \mathbb{Z}\right\}$$
$$\cup \left\{(2n+1)\pi - x, \quad x \in \left[\frac{\pi}{2}(4n+1), \frac{\pi}{2}(4n+3)\right] \mid n \in \mathbb{Z}\right\}$$

By observation, this function doesn't lend itself to being continuous. The most apparent thing is the variable m is negative when n is positive. This is just a case of notation, as m spans all integers. Hence if we write $m \to -n$, we have:

$$f(x) = \left\{-2n\pi + x, \quad x \in \left[\frac{\pi}{2}(4n-1), \frac{\pi}{2}(4n+1)\right] \mid n \in \mathbb{Z}\right\}$$
$$\cup \left\{(2n+1)\pi - x, \quad x \in \left[\frac{\pi}{2}(4n+1), \frac{\pi}{2}(4n+3)\right] \mid n \in \mathbb{Z}\right\}$$

In this form we can't do much, so we consider pieces, namely:

$$g : \left[\frac{\pi}{2}(4n-1), \frac{\pi}{2}(4n+3)\right], g(x) = \begin{cases} -2n\pi + x & x \leq \frac{\pi}{2}(4n+1) \\ (2n+1)\pi - x & x \geq \frac{\pi}{2}(4n+1) \end{cases}$$
$$= \frac{\pi}{2} - \left|x - \frac{\pi}{2}(4n+1)\right|$$

And subsequently rewrite $f(x)$ as:

$$f(x) = \left\{\frac{\pi}{2} - \left|x - \frac{\pi}{2}(4n+1)\right|, \quad x \in \left[\frac{\pi}{2}(4n-1), \frac{\pi}{2}(4n+3)\right] \mid n \in \mathbb{Z}\right\}$$

While derivation of this won't be given here (see site), using a sticking formula, one could write this, for $n \to \infty$, as:

$$f_n(x) = n\pi + \frac{1}{2}\sum_{k=-n}^{n}\left(\pi - \left|\left|x - \frac{\pi}{2}(4k-1)\right| - \left|x - \frac{\pi}{2}(4k+3)\right|\right|\right)$$

One could alternatively write this function in terms of the floor (or ceiling) function, provided each piece has a half-closed interval (this is a trivial assumption since the function is continuous; the floor function is not):

$$\begin{aligned} f(x) &= \left\{\frac{\pi}{2} - \left|x - \frac{\pi}{2}(4n+1)\right|, \quad x \in \left[\frac{\pi}{2}(4n-1), \frac{\pi}{2}(4n+3)\right] \mid n \in \mathbb{Z}\right\} \\ &= \frac{\pi}{2} - \left|x - \frac{\pi}{2}\left(1 + 4\left\{n, \quad x \in \left[\frac{\pi}{2}(4n-1), \frac{\pi}{2}(4n+3)\right) \mid n \in \mathbb{Z}\right\}\right)\right| \\ &= \frac{\pi}{2} - \left|x - \frac{\pi}{2}\left(1 + 4\left\{n, \quad \frac{2x}{\pi} + 1 \in [4n, 4n+4) \mid n \in \mathbb{Z}\right\}\right)\right| \\ &= \frac{\pi}{2} - \left|x - \frac{\pi}{2}\left(1 + 4\left\{n, \quad \frac{x}{2\pi} + \frac{1}{4} \in [n, n+1) \mid n \in \mathbb{Z}\right\}\right)\right| \\ &= \frac{\pi}{2} - \left|x - \frac{\pi}{2}\left(1 + 4\left\lfloor\frac{x}{2\pi} + \frac{1}{4}\right\rfloor\right)\right| \end{aligned}$$

Should one also be so inclined, using transformations of x and $f(x)$ (by properties of $\sin(x)$), we can rewrite this in terms of the mod function, related to the floor via $\operatorname{mod}(x, n) = x - n\left\lfloor\frac{x}{n}\right\rfloor$:

$$f(x) = \left|\operatorname{mod}(x - \frac{\pi}{2}, 2\pi) - \pi\right| - \frac{\pi}{2}$$

Furthermore (and I promise this is the last form), $f(x)$ can be approximated using a fourier series, using $g(x) \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$.

$$f(x) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{(2n-1)^2} \sin((2n-1)x) \right)$$